SOLUTIONS OF RADIALLY SYMMETRIC STRESS PROBLEMS BY NUMERICAL METHODS

BY

ELIO D'APPOLONIA

B.S., University of Alberta, 1942
M.S., University of Alberta, 1946

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN ENGINEERING
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INTRODUCTION

Solutions of stress problems have for the most part been obtained with the use of analytical methods which usually involve series expansions. Relatively little consideration has been given to the development of the numerical methods that are available for the solution of boundary value problems.

The mathematical difficulties in a formal solution very often make the practical engineering problem intractable. However the numerical methods, especially those which involve displacements, enable one to obtain a solution for most mixed boundary value problems.

When a body undergoes deformation due to some cause stresses may be produced throughout. The nature of these stresses is dependent on the disturbance, restraints, geometry of the body and physical properties of the material. Utilizing the theory of elasticity for elastic problems and of plasticity for inelastic problems, a solution of the stresses in a given problem can be obtained by solving under prescribed boundary conditions the differential equations which describe the state of deformation of the body. Since the solution of a large class of practical problems is very difficult by analytical treatment, the object of the investigation was to develop satisfactory approximate numerical methods for the solution of stress problems. The particular class of problems investigated possesses radial symmetry.

The solution to a stress problem is obtained at a discrete number of points. The system of coordinates used as a frame of reference and the size of the mesh or the network of points adopted in a solution depend on the nature of the problem. For many radially symmetric problems a cylindrical coordinate system using a square network of points in an axial

plane suffices. However, for certain problems which have sharp stress gradients due to large curvatures it is expedient to use an orthogonal curvilinear system of coordinates.

Two general methods were considered in the development of the numerical procedures for the solution of stress problems. One is the lattice analogy method in which it is imagined that the solid body can be replaced by a framework of bars. The second scheme involves the well known procedure of expressing a differential equation in its finite difference form. In both methods the displacements at the discrete points are considered, and a solution is obtained by a process of iteration.

The classical assumptions made in the theory of elasticity were also adopted in the development of the approximate methods for the solution of stresses in an elastic body.

Development And Results Of Methods Used For The Solution Of Elastic Problems

The lattice analogy method of solving elastic stress problems was considered first. The basic concepts in the development of the method are simple. It is assumed that the behaviour of the solid body under load can be approximated by a framework of bars subjected to the same load. The pattern or arrangement of bars is arbitrarily selected, but suitable to the type of problem to be solved. For the chosen lattice pattern the area of each bar is determined from a consideration of the deformation of an element of the solid body and the corresponding element of the framework, when both are subjected to similar load conditions. Or simply, the bar areas are so determined that the deformation of both elements, solid and lattice, are identical for certain simple force systems.

A consideration of statics at a node, for a system under

arbitrary displacements, enables one to develop the required displacement equations. These equations for a particular node are related to the force concentrations at the node in question and to the displacement components at the immediate surrounding nodes. Two such equations exist at all nodal points. The solution of the set of linear equations that occur in a problem is obtained by a process of successive corrections.

From the results of the displacement components at each of the various node points, forces in the bars of the framework are computed. By passing radial and axial sections through the center of a node and considering the bar forces on one side of a section, one is able to evaluate an equivalent uniform stress distribution for the solid body. This stress distribution is a step function and its approximation of the actual stress distribution in the solid body depends in part on the coarseness of the net used in solving the problem.

The lattice analogy method was found suitable for the solution of mixed boundary value problems. However, the method appears to be limited due to time consuming computational work to bodies whose geometric shape is formed of straight radial and axial lines. Curved surfaces in the framework are represented by a series of steps which follow in an approximate manner the outline of the boundary. Problems solved with such irregular boundaries in the analogy method showed high stress concentrations at the re-entrant corners. The solutions in these cases were not representative of the actual curved boundary problem. Solutions with finer subdivisions in the region of the curved surface were attempted, but the results did not appear to justify the lengthy computational work involved.

In an effort to obtain a more suitable method of defining a curved surface by a discrete number of points and of simplifying the solution of curved boundary problems to a point where a good approximation could be obtained with a coarse net in a short period of time, the lattice analogy method was replaced by the method of finite differences.

The basic equations to be integrated are the equations of statics (1) which are written in terms of the radial and axial component displacements u and w.

$$\left(\nabla^2 - \frac{1}{r^2}\right)u + \frac{1}{1-2\mu}\frac{\partial e_v}{\partial r} + \frac{K_r}{G} = o \qquad (a)$$

$$\nabla^2 w + \frac{1}{1-2\mu}\frac{\partial e_v}{\partial z} + \frac{K_z}{G} = o \qquad (b) \qquad (1)$$

In equations (1), the symbol e_v denotes the dilatation, μ and G are the elastic constants and K_r and K_z are components of the body force. Equations (1) are solved subject to the prescribed boundary conditions which may be stated in terms of stress, displacements, or a combination of both.

Derivatives at interior node points are defined by their simplest finite difference expressions. For example, a first derivative is specified by two points, second derivatives by three, and so forth. At the boundary fictitious node points were used and the derivatives were defined by their simplest finite difference expressions. Higher order terms in the finite difference expressions for a derivative were considered only in certain special cases of a curved boundary.

Boundary conditions specified in terms of stress are transformed into conditions of displacements by means of the stress-strain and strain-displacement relations. These conditions, when written in finite difference form permit one to determine the displacements at the fictitious node points in terms of known displacements and surface stresses.

Curved boundary surfaces in the numerical solution of a problem give rise to difficulties which are inherent in the adopted system of cylindrical coordinates. No general method appears to be suitable for establishing a curved boundary by a discrete number of points and of defining the boundary

conditions thereon. Actually, each problem has to be considered separately and the scheme used in solving a problem depends to a large extent on the computer's judgment and experience. Four allied schemes for treating curved boundaries were adopted. In the first scheme, slopes at a boundary were defined by two points. In the second scheme, four points were used and in the third, six. The fourth scheme dealt with multiple displacement values at the fictitious boundary nodes.

The various schemes were applied to the same notch problem. It was found that all four methods gave very nearly the same values for the axial stress at all points of the network. However, in the region of the root of the notch, neither the results obtained from the adopted coarse or fine subdivided nets agreed with the exact solution as given by Neuber in his book "Theory of Notch Stresses." The disagreement was primarily due to the coarseness of the net, which did not permit an accurate representation of the curved surface and the boundary conditions thereon. Further, the network of points in the region of the notch was not fine enough to furnish an accurate description of the stress variation.

To obtain a better solution for the notched bar problem a more convenient system of coordinates than the cylindrical system was adopted, namely, an orthogonal curvilinear system. Such a system has certain inherent properties which make it desirable for the solution of curved boundary problems. No difficulties arise in the delineation of a curved surface by a discrete number of points or in the determination of the finite difference expressions for the boundary conditions in terms of stress that apply to the curved surface.

The particular coordinate system used was the ellipsoidal. The network of intersecting traces of hyperboloids and ellipsoids on an axial plane yielded a pattern which had the desirable feature of the finer and denser subdivisions occuring in the region of high stress concentration at the root of the notch.

The solution of the notched bar problem in the curvilinear coordinate system was obtained subject to prescribed boundary displacements. The required boundary displacement components, well away from the root of the notch, were taken from a solution attained in a cylindrical coordinate system. Two solutions of the notched bar were made with a curvilinear pattern of points. One solution was secured with a network of 2 units by 2 units and the second from a network of 4 by 4 units. In the latter case, the axial stresses along the radial centerline were in close agreement with the exact solution. At the root of the notch, this solution was in error by -20.6 percent; whereas, for the same stress, the fine subdivided net in the cylindrical system was in error by -39.9 percent. By extrapolating from the two solutions made in the curvilinear system of coordinates, it was found that the axial stress at the root of the notch differed by -9.5 percent from the exact solution.

A brief note regarding body forces in the lattice analogy and finite difference methods is given in the concluding part of the thesis. The appendices treat the development of the lattice analogy method for the solution of two and three-dimensional elastic problems with reference to a rectangular system of coordinates.

Development And Results Of A Method Used For The Solution Of Inelastic Problems

Although the basic equations for a stress analysis in the plastic range are well known, mathematical complexities have limited the solutions of problems to only a few special cases. General methods of integrating the fundamental equations in the various theories of plasticity have not, as yet, been developed. To obtain solutions of mixed boundary value problems a numerical method was investigated.

The basic stress-strain equations used are of the deforma-

tion type which deal with small strains. These equations are considered as made up of two parts, one part dealing with the elastic deformations and the second with the plastic deformations. The yield and flow condition used is assumed to be a function only of the second invariant of the stress deviation. To define the start of plastic yielding or to define the flow condition for perfectly plastic materials the Huber-Mises-Hencky condition was used. In the development, it was assumed that the material was under a state of gradually increasing load, and the stress analysis was made at a particular stage during the loading process. Further, it was also assumed that: (a) the axes of principal stress and strain coincide; (b) the deviations of stress and strain are proportional; and (c) the intensity of stress is a completely determined function of the intensity of strain for all complex states of stress. The intensities of stress and strain, except for a constant factor, are equal to the square root of the second invariants of the stress and strain deviations, respectively.

The basic differential equations to be integrated were obtained by substituting stress-strain relations of the deformation type into the equations of statics. These differential equations were then written in terms of displacements. Further, the equations were split into two parts; one part was identical in form to equations (1), as obtained in the elastic theory. The second part involved a function associated with the inelastic behavior.

The integration of these differential equations subject to prescribed boundary conditions was obtained by first expressing them in their finite difference form and then solving the set of linear algebraic equations that result in a particular problem by a process of iteration. During each cycle or group of cycles in the iteration process, the part involving the inelastic function was considered constant, and for convenience it was considered to be a fictitious body force with components in the axial and radial directions. Since the fictitious body

forces and displacements are interrelated, changes in one produced a change in the other. Thus, iterations were carried out until both the displacements and the fictitious body forces became stabilized.

The above procedure was applied to a simple problem of a thick hollow cylinder subjected to an internal pressure and to two other more complex problems. These were (a) a short cylinder subjected to a tensile parabolic stress distribution over its ends; (b) a short cylinder compressed between rough rigid surfaces. The solution for the thick hollow cylinder problem was compared to existing solutions in which the assumption of incompressibility was not made. A comparison of the stresses was very good, with variations of less than 1.0 percent in both the elastic and plastic ranges.

Conclusions

The lattice analogy method as developed is suitable for the solution of mixed boundary value problems in which the geometric shape of the body is formed of straight radial and axial lines. Curved boundaries impose a limitation on the method, as the framework can only approximate the curved boundary by a series of steps. At each re-entrant corner the high stress concentrations that appear create a stress disturbance over a large region of the framework. The use of fine subdivided frameworks in the region of the curved surface to localize the effect of each re-entrant corner is not desirable as the computational work in a solution becomes exceedingly great and the convergence very slow.

The finite difference method appears to be suitable for the solution of certain curved boundary problems when cylindrical coordinates are used as the frame of reference. This system of coordinates is adequate to determine a stress distribution provided the geometrical shape of the solid body is such that no regions of exceedingly sharp stress gradient occur. For other problems, networks of fine subdivisions in regions

of large curvature are not sufficient to enable one to obtain a solution which would be indicative of that of the actual problem. In these cases it is necessary to perform the numerical work with reference to a suitable orthogonal system of coordinates.

The numerical method presented for the solution of inelastic problems deals with stress-strain relations of the deformation type. The flow condition is considered to be a function only of the second invariant of the stress deviation. The basic equations to be integrated that arise from these fundamental relations and the equations of statics are linear in form when expressed in terms of displacement components. This characteristic of linearity of the differential equations was a desirable feature, as it permited one to obtain a solution of an inelastic problem by solving a sequence of elastic problems through a process of successive corrections.

The utility of the finite difference methods presented for the solution of certain elastic and inelastic problems is beyond question. However there still remains the development of means of hastening the convergence of a problem to a solution, also of developing systematic methods of group corrections to shorten the computational labour. Further investigations and improvements are also required in the definition of a curved surface and the boundary conditions thereon by a rectangular pattern of a discrete number of points.

The lattice analogy method in its present form is not suitable for the solution of inelastic problems. This phase of the method requires further investigation and development. Also the numerical methods presented require consideration with a view to solving anisotropic problems for both the elastic and inelastic behavior.